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Supersymmetry and Euler multiplets

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Abstract: Some massless supermultiplets appear as the trivial solution of Kostant's equation, a Dirac-like equation over special cosets. We study two examples; one over the coset $\mathrm{SU}(3)/\mathrm{SU}(2)\times\mathrm{U}(1)$ contains the $N=2$ hypermultiplet in $(3+1)$ dimensions with U(1) as helicity; the other over the coset $F_4/\mathrm{SO}(9)$ describes the $N=1$ supermultiplet in eleven dimensions, where SO(9) is the light-cone little group. We present the general solutions to Kostant's equation for both cases; they describe massless physical states of arbitrary spins which display the same relations as the fields in the supermultiplets. They come in sets of three representations called Euler triplets, but do not display supersymmetry although the number of bosons and fermions is the same when spin-statistics is satisfied. We build the free light-cone lagrangian for both cases.

Keywords: Field Theories in Higher Dimensions, M-Theory, Space-Time Symmetries, Supergravity Models.

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Contents

1. Introduction

String theory has given us an understanding of the dimension of spacetime as a crucial concept for building consistent quantum mechanical theories. This insight has been further strengthened with the introduction of supersymmetry, since the spinor depends not only on the spacetime dimensions but also on the possibility to implement the Majorana and Weyl conditions. The little group for the superstring is SO(8), one of the most beautiful and unique groups. Its Dynkin diagram is the famous Mercedes symbol, and the group is really the Mercedes of the orthogonal groups. It has a triality symmetry and its three eight-dimensional vector, spinor and cospinor representations are readily interchangeable. In the light-cone formulation of the superstring it is these three representations that build up the theory and in quantum calculations there are marvellous cancellations between the bosonic and fermionic contributions that render the theory perturbatively finite.

From this viewpoint the emergence of M-theory as an even more general theory was unexpected. Eleven-dimensional space is singled out as the maximal one to carry a supermultiplet with the graviton as the highest spin field, but is there some group-theoretic reasons why an eleven-dimensional spacetime should make special sense? The little group SO(9) has a nondescript Dynkin diagram. It displays no symmetry and no cars use it as a symbol, not even the Trabant.

In this paper, we show that SO(9) does have hidden attributes. They come in the form of a wealth of internal relations between sets of its representations, again making it a plausible candidate for a fundamental symmetry. It was found [1] that some of its irreps naturally group together into triplets which are such that bosonic and fermionic degrees of freedom match up the same way as they do in eleven-dimensional supergravity. This makes the triplets interesting not only from a mathematical point of view but also from a quantum physical one. Higher order loop calculations in a supersymmetric theory have huge cancellations between fermionic and bosonic contributions since their contributions can be written in terms of group-theoretic indices related to the little group which match up. In eleven dimensional supergravity, the matching among its three fields is not total, but fails in the highest eighth order invariant, which produces a nonrenormalizabledivergence in high loop order [2]. Each Euler triplet displays the same partial matching among its representations. It is our hope that total matching can be restored only after we sum over an infinite number of Euler triplets. This would entail the vanishing of an infinite sum of positive numbers.

All other Euler triplets involve higher spin massless fields. This is reminiscent of a problem in string theory which has not been solved, namely the limit of infinite Regge slope, i.e. the zero-tension limit, where one expects all states to become massless, with an infinite number of states for each spin. However this limit is not tractable at the amplitude level. Can we connect it to a theory of massless spins in higher dimensions?

An interesting question in this context is what happens to the symmetries of a massive theory when all particles become massless. Usually the linearly realized symmetry is enhanced as it happens when a spontaneously broken symmetry is restored. The gauge invariance is of course restored but what happens to the supersymmetry? In a previous paper [3], we have shown that a continuous spin representation of the SuperPoincaré group connects massless states of arbitrarily high spins. Another alternative, the use of a tower of Euler triplets, seems to require much bigger supersymmetries. This issue will be discussed in this paper.

Since the discovery of the triplets there has been great progress in the mathematical understanding of their beautiful properties [4]. They arise for embeddings where both group and subgroup have the same rank. In the case above SO(9), alias B_4, is a subgroup of F_4 with the same rank, and the quotient space F_4/B_4 has Euler number three giving a triplet of SO(9) to every irrep of F_4. We have elsewhere [5, 6, 7] listed the cases with up to 16-dimensional cosets and in this list we find multiplets with the above properties, extending the multiplets of besides the 11-dimensional supergravity also $N = 8$ supergravity, $N = 4$ Yang-Mills and the $N = 2$ hypermultiplet. All these Euler triplets arise as solutions of Kostant's equation [8], which is a Dirac-like equation on the coset.

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It is quite interesting that the exceptional algebra F_4 enters into the description of an 11-dimensional theory. We have seen the exceptional groups emerge as gauge groups and there is a direct line from the gauge group of the Standard Model via $\mathrm{SU}(5) = E_4$ and $\mathrm{SO}(10) = E_5$ up to the ultimate exceptional group E_8, the Delahaye of the groups. (Rolls Royce for non-French readers.) So far there has not been any trace of exceptional groups extending the space time symmetry. There is a simple reason for this since they relate tensor and spinor representations of their orthogonal subgroups, while spin statistics treat them differently. However, the exceptional groups are the most unique and beautiful ones and it is many physicists' dream that they represent the ultimate symmetry of the world.

One question that will arise is if we should use all triplets as a candidate theory or if there is a natural selection among them. In this report we show a method where we define a superfield which naturally has an equal number of bosonic and fermionic degrees of freedom and is a natural extension of the superfield containing the smallest multiplet. We will use a light-cone frame formulation which will tie together the superspace with the external symmetry.

We start by discussing in some detail the simpler case of $\mathrm{SU}(3)/\mathrm{SU}(2) \times \mathrm{U}(1)$, which we interpret as a model for higher spin massless fields in four dimensions, generalizing the $N = 2$ hypermultiplet. We then apply the same methods to the more complicated case of more direct physical interest where the Euler triplets appear as a natural generalization of supergravity in eleven dimensions.

2. Euler triplets for $\mathrm{SU}(3)/\mathrm{SU}(2) \times \mathrm{U}(1)$

We now present a detailed analysis of the Euler triplets associated with the coset $\mathrm{SU}(3)/\mathrm{SU}(2) \times \mathrm{U}(1)$. There are infinitely many of them, one for each representation of $\mathrm{SU}(3)$. The trivial solution describes the light-cone degrees of freedom of the $N = 2$ hypermultiplet in four dimensions, with $\mathrm{U}(1)$ interpreted as helicity. Hence we begin by reminding the reader of the well-known light-cone description of that multiplet.

2.1 The $N = 2$ hypermultiplet in 4 dimensions

The massless $N = 2$ scalar hypermultiplet contains two Weyl spinors and two complex scalar fields, on which the $N = 2$ SuperPoincaré algebra is realized. Introduce the light-cone hamiltonian

$$P^- = \frac{p\overline{p}}{p^+}\,, \tag{2.1}$$

where $p = \frac{1}{\sqrt{2}}(p^1 + ip^2)$. The front-form supersymmetry generators satisfy the anticommutation relations

$$\begin{aligned}
\{\mathcal{Q}^m_+, \overline{\mathcal{Q}}^n_+\} &= -2\delta^{mn} p^+\,, \\
\{\mathcal{Q}^m_-, \overline{\mathcal{Q}}^n_-\} &= -2\delta^{mn} \frac{p\overline{p}}{p^+}\,, \qquad m,n = 1,2\,, \\
\{\mathcal{Q}^m_+, \overline{\mathcal{Q}}^n_-\} &= -2\delta^{mn} p\,.
\end{aligned} \tag{2.2}$$

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The kinematic supersymmetries are expressed as

$$\mathcal{Q}_{+}^{m} = -\frac{\partial}{\partial \overline{\theta}^{m}} - \theta_m p^+ \,, \qquad \overline{\mathcal{Q}}_{+}^{m} = \frac{\partial}{\partial \theta^{m}} + \overline{\theta}_m p^+ \,, \tag{2.3}$$

while the kinematic Lorentz generators are given by

$$\begin{aligned} M^{12} &= i(x\overline{p} - \overline{x}p) + \frac{1}{2}\theta_m \frac{\partial}{\partial \theta_m} - \frac{1}{2}\overline{\theta}^m \frac{\partial}{\partial \overline{\theta}^m} \,, \\ M^{+-} &= -x^- p^+ - \frac{i}{2}\theta_m \frac{\partial}{\partial \theta_m} - \frac{i}{2}\overline{\theta}^m \frac{\partial}{\partial \overline{\theta}^m} \,, \\ M^{+} &\equiv \frac{1}{\sqrt{2}}(M^{+1} + iM^{+2}) = -xp^+ \,, \qquad \overline{M}^{+} = -\overline{x}p^+ \,, \end{aligned} \tag{2.4}$$

where $x = \frac{1}{\sqrt{2}}(x^1 + ix^2)$, and where the two complex Grassmann variables satisfy the anticommutation relations

$$\begin{aligned} \left\{\theta_m, \frac{\partial}{\partial \theta_n}\right\} &= \left\{\overline{\theta}^m, \frac{\partial}{\partial \overline{\theta}^n}\right\} = \delta^{mn} \,, \\ \left\{\theta_m, \frac{\partial}{\partial \overline{\theta}^n}\right\} &= \left\{\overline{\theta}^m, \frac{\partial}{\partial \theta_n}\right\} = 0 \,. \end{aligned}$$

The (free) hamiltonian-like supersymmetry generators are simply

$$\mathcal{Q}_{-}^{m} = \frac{\overline{p}}{p^+}\mathcal{Q}_{+}^{m} \,, \qquad \overline{\mathcal{Q}}_{-}^{m} = \frac{p}{p^+}\overline{\mathcal{Q}}_{+}^{m} \,, \tag{2.5}$$

and the light-cone boosts are given by

$$\begin{aligned} M^{-} &= x^- p - \frac{1}{2}\{x, P^-\} + i\frac{p}{p^+}\theta_m \frac{\partial}{\partial \theta_m} \,, \\ \overline{M}^{-} &= x^- \overline{p} - \frac{1}{2}\{\overline{x}, P^-\} + i\frac{\overline{p}}{p^+}\overline{\theta}^m \frac{\partial}{\partial \overline{\theta}^m} \,. \end{aligned} \tag{2.6}$$

This representation of the superPoincaré algebra is reducible, as it can be seen to act on reducible superfields $\Phi(x^-, x^i, \theta_m, \overline{\theta}^m)$, because the operators

$$\mathcal{D}_{+}^{m} = \frac{\partial}{\partial \overline{\theta}^m} - \theta_m p^+ \,, \tag{2.7}$$

anticommute with the supersymmetry generators. As a result, one can achieve irreducibility by acting on superfields for which

$$\mathcal{D}_{+}^{m}\, \Psi = \left[\frac{\partial}{\partial \overline{\theta}^m} - \theta_m p^+\right]\Psi = 0 \,, \tag{2.8}$$

solved by the chiral superfield

$$\Psi\left(y^-, x^i, \theta_m\right) = \psi_0\left(y^-, x^i\right) + \theta_m \psi^m\left(y^-, x^i\right) + \theta_1\theta_2\psi^{12}\left(y^-, x^i\right) \,. \tag{2.9}$$

The field entries of the scalar hypermultiplet now depend on the combination

$$y^- = x^- - i\theta_m\overline{\theta}^m \,, \tag{2.10}$$

and the transverse variables. Acting on this chiral superfield, the constraint is equivalent to requiring that

$$\mathcal{Q}^m_+ \approx -2p^+\theta_m\,, \qquad \overline{\mathcal{Q}}^m_+ \approx \frac{\partial}{\partial\theta_m}\,, \tag{2.11}$$

where the derivative is meant to act only on the naked θ_m's, not on those hiding in y^-. This light-cone representation is well-known, but we repeat it here to set our conventions and notations.

2.2 Kostant's equation

Let T^A , $A = 1, 2, \ldots, 8$, denote the SU(3) generators. Its SU(2) × U(1) subalgebra is generated by T^i, $i = 1, 2, 3$, and T^8. Introduce Dirac matrices over the coset

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}\,,$$

for $a, b = 4, 5, 6, 7$, to define the Kostant equation over the coset SU(3)/ SU(2) × U(1) as

$$\not{\mathcal{K}}\,\Psi = \sum_{a=4,5,6,7} \gamma^a T_a \Psi = 0\,. \tag{2.12}$$

The Kostant operator commutes with the SU(2) × U(1) generators

$$L_i = T_i + S_i\,,\ i = 1, 2, 3\,; \qquad L_8 = T_8 + S_8\,, \tag{2.13}$$

sums of the SU(3) generators and of the "spin" part, expressed in terms of the γ matrices as

$$S_j = -\frac{i}{4} f_{jab}\gamma^{ab}\,, \qquad S_8 = -\frac{i}{4} f_{8ab}\gamma^{ab}\,, \tag{2.14}$$

where $\gamma^{ab} = \gamma^a\gamma^b\,, a \neq b$, and $f_{jab}\,, f_{8ab}$ are structure functions of SU(3).

The Kostant equation has an infinite number of solutions which come in groups of three representations of SU(2) × U(1), called Euler triplets. For each representation of SU(3), denoted by $[a_1\, a_2]$, there is a unique Euler triplet $\{a_1, a_2\}$, made up of three representations of SU(2) × U(1)

$$\{a_1, a_2\} \;\equiv\; [a_2]_{-\frac{2a_1+a_2+3}{6}} \oplus [a_1 + a_2 + 1]_{\frac{a_1-a_2}{6}} \oplus [a_1]_{\frac{2a_2+a_1+3}{6}}\,,$$

where a_1, a_2 are the Dynkin labels of the associated SU(3) representation. Here, $[a]$ stands for the $a = 2j$ representation of SU(2), and the subscript denotes the U(1) charge. The Euler triplet corresponding to $a_1 = a_2 = 0$,

$$\{0, 0\} \;=\; [0]_{-\frac{1}{2}} \oplus [1]_0 \oplus [0]_{\frac{1}{2}}\,,$$

describes the degrees of freedom of the $N = 2$ supermultiplet, where the properly normalized U(1) is interpreted as the helicity of the four-dimensional Poincaré algebra.

Below, we wish to explore the possibility of linking this supersymmetric triplet to those for which $a_{1,2} \neq 0$, while preserving at least relativistic invariance. Of particular interest will be the algebraic operations that link the different Euler triplets. Their use will enable us to define supersymmetry-like operations acting on the higher Euler triplets, which serve as the shadow of the light-cone supersymmetry of the lowest Euler triplet.

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The U(1) charges of the higher triplets are generally rational numbers, which means that they display parastatistics, but the triplets for which

$$a_1 = a_2 \mod(3)\,,$$

contain half-odd integer or integer U(1) charges, and satisfy Fermi-Dirac statistics. A self-conjugate subset

$$\{a, a\}\ :\quad [a]_{-\frac{a+1}{2}} \oplus [2a+1]_0 \oplus [a]_{\frac{a+1}{2}}\ ,$$

contains equal number of half-odd integer-helicity fermions and integer-helicity bosons, and satisfy CPT. As the helicity gap between the representations increases indefinitely in half integer steps, the symmetry operations that relate its members have helicities $\pm(a+1)/2$. When $a = 0$, they can be identified with the usual supersymmetries, and they are fermionic as long as a is even. The others are generated from complex representations with $N = 1, 2, \ldots$ like

$$\{a, a+3N\}\ :\quad [a]_{-\frac{a+N+1}{2}} \oplus [2a+3N+1]_{\frac{N}{2}} \oplus [a+3N]_{\frac{a+N+1}{2}}\ .$$

The helicity gap also increases in half integer steps, starting at one-half. Since CPT requires states of opposite helicity, these must be accompanied by their conjugates, $\{a + 3N, a\}$, with all helicities reversed.

A special case deserves consideration: when $a = 0$, the helicity gap can be as small as 1/2, like the regular supersymmetry. The simplest example is

$$\{0, 3\}\ :\quad [0]_{-1} \oplus [4]_{\frac{1}{2}} \oplus [3]_1\ ,$$

where the helicity gap is 3/2 and 1/2. When we add the CPT conjugate

$$\{3, 0\}\ :\quad \oplus[0]_1 \oplus [4]_{-\frac{1}{2}} \oplus [3]_{-1}\ ,$$

we end up with states separated by half a unit of helicity as in the supersymmetric multiplets. As they occur in different representations of SU(2), equality between bosons and fermions is achieved only after including the CPT conjugate, but as long as SU(2) remains unbroken, relativistic supersymmetry cannot be implemented on these states. The case $N = 2$ yields states of helicity 1 and 3/2, and $N = 3$ contains eleven states of helicity 2, and so on.

It appears that while Poincaré symmetry can be implemented on an infinite subset of Euler triplets, relativistic supersymmetry can be realized on a finite subset, and only after the SU(2) is broken. In particular the need for operators that shift helicity by more than half units makes it unlikely that a relativistic supersymmetric theory of Euler triplets can be found.

In addition, the higher Euler triplets include states with helicities larger than 2, which cannot be interpreted as massive relativistic states since they do not arrange themselves in SO(3) representations. A relativistic description requires them to be massless particles of spin higher than 2 in four dimensions.

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There are well-known difficulties with such theories [9]; in the flat space limit they must decouple from the gravitational sector, although this can be circumvented in curved space [10], or if there is an infinite number of such particles. Our purpose is to investigate if a relativistic theory can be formulated with an infinite number of Euler multiplets, in which a light-cone version of a new type of space-time fermionic symmetry is present.

2.3 Grassmann numbers and Dirac matrices

We will express the solutions of Kostant's equations as chiral superfields over space-time and internal variables pertaining to SU(3). For that purpose, we first identify the spin part of the U(1) generator S_8 with the spin part in Eq. (2.5) taking the condition (2.8) into account, thus writing the S_i in terms of $\theta's$. An appropriate representation is then

$$\gamma^4 + i\gamma^5 = i\sqrt{\frac{2}{p^+}}\mathcal{Q}^1_+\,, \qquad \gamma^4 - i\gamma^5 = i\sqrt{\frac{2}{p^+}}\overline{\mathcal{Q}}^1_+\,, \tag{2.15}$$

$$\gamma^6 + i\gamma^7 = i\sqrt{\frac{2}{p^+}}\mathcal{Q}^2_+\,, \qquad \gamma^6 - i\gamma^7 = i\sqrt{\frac{2}{p^+}}\overline{\mathcal{Q}}^2_+\,, \tag{2.16}$$

in terms of the kinematic $N = 2$ light-cone supersymmetry generators defined in the previous section. We can check that S_8 indeed agrees with the spin part of Eq. (2.5) (after proper normalization). As the Kostant operator anticommutes with the constraint operators

$$\{\not{\mathcal{K}},\, \mathcal{D}^m_+\} = 0\,, \tag{2.17}$$

its solutions can be written as chiral superfields, on which the γ's become

$$\gamma^4 + i\gamma^5 = -2i\sqrt{2p^+}\,\theta_1\,, \qquad \gamma^4 - i\gamma^5 = i\sqrt{\frac{2}{p^+}}\,\frac{\partial}{\partial\theta_1}\,, \tag{2.18}$$

$$\gamma^6 + i\gamma^7 = -2i\sqrt{2p^+}\,\theta_2\,, \qquad \gamma^6 - i\gamma^7 = i\sqrt{\frac{2}{p^+}}\,\frac{\partial}{\partial\theta_2}\,, \tag{2.19}$$

The complete "spin" parts of the SU(2)×U(1) generators, expressed in terms of Grassmann variables, do not depend on p^+,

$$\begin{aligned} S_1 &= \frac{1}{2}\left(\theta_1\frac{\partial}{\partial\theta_2} + \theta_2\frac{\partial}{\partial\theta_1}\right), \qquad & S_2 &= -\frac{i}{2}\left(\theta_1\frac{\partial}{\partial\theta_2} - \theta_2\frac{\partial}{\partial\theta_1}\right), \\ S_3 &= \frac{1}{2}\left(\theta_1\frac{\partial}{\partial\theta_1} - \theta_2\frac{\partial}{\partial\theta_2}\right), \qquad & S_8 &= \frac{\sqrt{3}}{2}\left(\theta_1\frac{\partial}{\partial\theta_1} + \theta_2\frac{\partial}{\partial\theta_2} - 1\right). \end{aligned} \tag{2.20}$$

Using Grassmann properties, the SU(2) Casimir operator can be written as

$$\vec{S}^2 = \frac{3}{4}\left(\theta_1\frac{\partial}{\partial\theta_1} - \theta_2\frac{\partial}{\partial\theta_2}\right)^2; \tag{2.21}$$

it has only two eigenvalues, 3/4 and zero. These SU(2) generators obey a simple algebra

$$S_i\,S_j = \frac{1}{3}\vec{S}\cdot\vec{S}\delta_{ij} + \frac{i}{2}\epsilon_{ijk}S_k\,. \tag{2.22}$$

The helicity, identified with S_8 up to a normalizing factor of $\sqrt{3}$, leads to half-integer helicity values on the Grassmann-odd components of the (constant) superfield representing the hypermultiplet.

2.4 Solutions of Kostant's equation

Consider now Kostant's equation over $\mathrm{SU}(3)/\,\mathrm{SU}(2)\times\mathrm{U}(1)$. It is given by

$$\not{\mathcal{K}}\,\Psi \;=\; \sum_{a=4,5,6,7} \gamma^a T_a\,\Psi = 0\,,$$

where Ψ is now a chiral superfield as in (9) extended to contain internal group variables. Expanding the solutions and the Dirac matrices in terms of Grassmann variables yields two independent pairs of equations

$$(T_4+iT_5)\psi_1+(T_6+iT_7)\psi_2=0\,;\qquad (T_4-iT_5)\psi_2-(T_6-iT_7)\psi_1=0\,,$$

and

$$(T_4-iT_5)\psi_0-(T_6+iT_7)\psi_{12}=0\,;\qquad (T_6-iT_7)\psi_0+(T_4+iT_5)\psi_{12}=0\,,$$

that is in the representation of the generators given in Appendix A

$$(z_1\partial_3-\overline{z}_3\overline{\partial}_1)\psi_1+(z_2\partial_3-\overline{z}_3\overline{\partial}_2)\psi_2=0\,;\qquad (z_3\partial_1-\overline{z}_1\overline{\partial}_3)\psi_2-(z_3\partial_2-\overline{z}_2\overline{\partial}_3)\psi_1=0\,,$$

$$(z_3\partial_1-\overline{z}_1\overline{\partial}_3)\psi_0-(z_2\partial_3-\overline{z}_3\overline{\partial}_2)\psi_{12}=0\,;\qquad (z_3\partial_2-\overline{z}_2\overline{\partial}_3)\psi_0+(z_1\partial_3-\overline{z}_3\overline{\partial}_1)\psi_{12}=0\,.$$

The homogeneity operators

$$D=z_1\partial_1+z_2\partial_2+z_3\partial_3\,,\qquad \overline{D}=\overline{z}_1\overline{\partial}_1+\overline{z}_2\overline{\partial}_2+\overline{z}_3\overline{\partial}_3$$

commute with $\not{\mathcal{K}}$, allowing the solutions of Kostant equation to be arranged in terms of homogeneous polynomials, on which a_1 is the eigenvalue of D and a_2 that of $\overline{D}$. The solutions can also be labeled in terms of the $\mathrm{SU}(2)\times\mathrm{U}(1)$ generated by the operators

$$L_i=T_i+S_i\,,\quad i=1,2,3\,;\qquad L_8=T_8+S_8\,.$$

The solutions for each triplet, are easily written for the highest weight states of each representation,

$$\begin{aligned}\Psi \;=\; & z_3^{a_1}\overline{z}_2^{a_2}+ \qquad \text{labels}[a_2]_{-\frac{2a_1+a_2+3}{6}}\,,\\ & +\theta_1 z_1^{a_1}\overline{z}_2^{a_2}+ \qquad \text{labels}[a_1+a_2+1]_{\frac{a_1-a_2}{6}}\,,\\ & +\theta_1\theta_2\; z_1^{a_1}\;\overline{z}_3^{a_2}\,, \qquad \text{labels}[a_1]_{\frac{2a_2+a_1+3}{6}}\,,\end{aligned} \tag{2.23}$$

where $[\ldots]$ are the SU(2) Dynkin labels. All other states are obtained by repeated action of the lowering operator

$$L_1-iL_2=\theta_2\frac{\partial}{\partial\theta_1}+(z_2\partial_1-\overline{z}_1\overline{\partial}_2)\,,$$

giving us all the states within each Euler triplet. Every state has finally to be multiplied by a field depending on the coordinates y^- and x^i. We see that in this notation, each triplet differs from the next by the degree of homogeneity. This implies that it is possible to move across the triplets by simple multiplication or differentiation. This is the object of the next section.

2.5 Relating the triplets-shadow supersymmetry

Since all Euler triplet superfields can be written in terms of components which are homogeneous polynomials of order a_1 in the variables $\mathbf{z}$ and a_2 in the $\overline{\mathbf{z}}$, we can easily construct algebraic operations which, when applied to one Euler triplet, generate another. These can then be used to define new operations called *shadow supersymmetry*, which act as ladder operators between the components of a given higher Euler triplet. They generalize the familiar supersymmetry operations to the higher Euler triplets, but cannot be upgraded to space-time supersymmetry.

Introduce the projection operators onto the components of the chiral superfields:

$$\begin{aligned} P_0 &= (1-\mathcal{P}_1)(1-\mathcal{P}_2)\,, \qquad & P_1 &= \mathcal{P}_1(1-\mathcal{P}_2)\,, \\ P_2 &= \mathcal{P}_2(1-\mathcal{P}_1)\,, & P_{12} &= \mathcal{P}_1\mathcal{P}_2\,, \end{aligned}$$

where $\mathcal{P}_i \equiv \theta_i\partial/\partial\theta_i$, $i=1,2$, and which satisfy the relations

$$P_a P_b = \delta_{ab} P_a\,. \tag{2.24}$$

There are two basic ladder operators which increase a_1 and a_2 by one unit, respectively. Let us begin by constructing the spectrum-generating ladder operators which generate $\Psi_{\{a_1+1,a_2\}}$ when applied to $\Psi_{\{a_1,a_2\}}$. Here we limit our construction of these operations to the highest weight states of each Euler triplet, and refer the reader to Appendix B for a more general presentation.

Since the non-Grassmann component for each triplet differs simply by the power in z_3, we start with the operator

$$\mathcal{A}^\dagger \;\equiv\; z_3\, P_0\,, \tag{2.25}$$

which multiplies the lowest component of any superfield by z_3. An SU(2) singlet with $-1/3$ helicity, it acts as a raising operator between Euler triplets. The corresponding lowering operator is

$$\mathcal{A} \;\equiv\; P_0 \frac{\partial}{\partial z_3}\,. \tag{2.26}$$

These two operators act as harmonic oscillators on the lowest superfield component

$$\left[\mathcal{A}, \mathcal{A}^\dagger\right] = P_0\,. \tag{2.27}$$

The highest weight θ-dependent terms of two adjacent Euler triplets differ by a single power of z_1. This leads us to the operators

$$\mathcal{B}^\dagger \;\equiv\; z_1\,(1-P_0)\,, \qquad \mathcal{B} \;\equiv\; \frac{\partial}{\partial z_1}\,(1-P_0)\,,$$

which act as harmonic oscillators on the θ-dependent terms

$$\left[\mathcal{B}, \mathcal{B}^\dagger\right] = (1-P_0)\,. \tag{2.28}$$

It follows that the action of $\mathcal{A}^\dagger$ and $\mathcal{B}^\dagger$ on the highest weight components of $\Psi_{\{a_1,a_2\}}$ generates the highest weights of the $\Psi_{\{a_1+1,a_2\}}$ Euler triplet. All other states are obtained by the action of the lowering operators.

A similar construction for the ladder operators which change a_2 by one unit leads us to the operators

$$\overline{\mathcal{A}}^\dagger \equiv P_{12}\bar{z}_3\,, \qquad \overline{\mathcal{A}} \equiv P_{12}\frac{\partial}{\partial \bar{z}_3} \tag{2.29}$$

and

$$\overline{\mathcal{B}}^\dagger \equiv (1-P_{12})\,\overline{z}_2\,, \qquad \overline{\mathcal{B}} \equiv (1-P_{12})\frac{\partial}{\partial \bar{z}_2}\,, \tag{2.30}$$

which act as harmonic oscillators on the appropriate superfield components.

The action of kinematic supersymmetry among the highest weight components is simply multiplication and differentiation by θ_i. However with the ladder operations we have just introduced, there is another way to move within each triplet.

For example on the $\Psi_{\{1,0\}}$ superfield, we can use the ladder operator to step down once to the hypermultiplet, perform a supersymmetry operation and then step up once again. This operation, called *shadow supersymmetry*, is generated by

$$\mathcal{Q}_1^{\downarrow} \equiv \mathcal{B}^\dagger\theta_1\mathcal{A}\,, \qquad \mathcal{Q}_2^{\downarrow} \equiv \mathcal{B}^\dagger\theta_2\mathcal{B}\,. \tag{2.31}$$

The inverse operations will be

$$\mathcal{Q}_1^{\uparrow} \equiv \mathcal{A}^\dagger\frac{\partial}{\partial\theta_1}\mathcal{B}\,, \qquad \mathcal{Q}_1^{\uparrow} \equiv \mathcal{B}^\dagger\frac{\partial}{\partial\theta_2}\mathcal{B}\,. \tag{2.32}$$

Properly normalized, these anticommute to one on the $\{1,0\}$ superfield. Generalizing to the $\{a_1,a_2\}$ Euler triplet, we can define similar operators

$$\mathcal{Q}_1^{\downarrow[a_1,a_2]} \equiv \left(\mathcal{B}^\dagger\right)^{a_1}\left(\overline{\mathcal{B}}^\dagger\right)^{a_2}\theta_1\,(\mathcal{A})^{a_1}\,\left(\overline{\mathcal{B}}\right)^{a_2}\,, \tag{2.33}$$

$$\mathcal{Q}_2^{\downarrow[a_1,a_2]} \equiv \left(\mathcal{B}^\dagger\right)^{a_1}\left(\overline{\mathcal{A}}^\dagger\right)^{a_2}\theta_2\,(\mathcal{B})^{a_1}\,\left(\overline{\mathcal{B}}\right)^{a_2}\,, \tag{2.34}$$

together with their respective raising counterparts. The operators form an anticommuting algebra as they manifestly close on the highest weight components of the Euler triplets, and generalize kinematic supersymmetry. However unlike the usual supersymmetry which change helicity by half a unit, these new operations carry arbitrarily high values of helicity and cannot be used to construct space-time supersymmetry.

In this section, we have considered the baby example associated with the SU(3)/SU(2)$\times$U(1) coset, in order to gain familiarity with its Euler triplets. We have built its explicit Euler triplet solutions, and shown how they are related to each other. While the triplets show no space-time supersymmetry, we have been able to relate its components by nilpotent operations which serve as a generalization of kinematic supersymmetry to the higher triplets. In the next section, we apply the same techniques to the more complicated but much more interesting case of $F_4/\,\mathrm{SO}(9)$.

3. Supergravity in eleven dimensions

The ultimate field theory without gravity is the finite $N=4$ Super Yang-Mills theory in four dimensions [11]. Eleven dimensional $N=1$ Supergravity [12], the ultimate field theory with gravity, is not renormalizable; it does not stand on its own as a physical theory, as far as we know today. However, the eleven-dimensional theory has been recently revived as the infrared limit of the presumably finite M-theory which, like characters on the walls of Plato's cave, has revealed itself only through its compactified versions onto lower-dimensional manifolds.

3.1 Superalgebra

$N = 1$ supergravity in eleven dimension is a local field theory that contains three massless fields, the familiar symmetric second-rank tensor, $h_{\mu\nu}$ which represents gravity, a three-form field $A_{\mu\nu\rho}$, and the Rarita-Schwinger spinor $\Psi_{\mu\,\alpha}$. From its lagrangian, one can derive the expression for the super Poincaré algebra, which in the light cone gauge assumes the particularly simple form in terms of the nine (16×16) γ_i matrices which form the Clifford algebra

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}\,, \qquad i, j = 1, \ldots, 9\,.$$

Supersymmetry is generated by the sixteen real supercharges

$$\mathcal{Q}^a_{\pm} = \mathcal{Q}^{a*}_{\pm}\,,$$

which satisfy

$$\begin{aligned} \left\{\mathcal{Q}^a_+, \mathcal{Q}^b_+\right\} &= \sqrt{2}\,p^+\delta^{ab}\,, \\ \left\{\mathcal{Q}^a_-, \mathcal{Q}^b_-\right\} &= \frac{\vec{p}\cdot\vec{p}}{\sqrt{2}p^+}\delta^{ab}\,, \\ \left\{\mathcal{Q}^a_+, \mathcal{Q}^b_-\right\} &= -(\gamma_i)^{ab}p^i\,, \end{aligned} \tag{3.1}$$

and transform as Lorentz spinors

$$\left[M^{ij}, \mathcal{Q}^a_{\pm}\right] = \frac{i}{2}(\gamma^{ij}\mathcal{Q}_{\pm})^a\,, \qquad \left[M^{+-}, \mathcal{Q}^a_{\pm}\right] = \pm\frac{i}{2}\mathcal{Q}^a_{\pm}\,, \tag{3.2}$$

$$\left[M^{\pm i}, \mathcal{Q}^a_{\mp}\right] = 0\,, \qquad \left[M^{\pm i}, \mathcal{Q}^a_{\mp}\right] = \pm\frac{i}{\sqrt{2}}(\gamma^i\mathcal{Q}_{\pm})^a\,. \tag{3.3}$$

A very simple representation of the 11-dimensional super-Poincaré generators can be constructed, in terms of sixteen anticommuting real χ's and their derivatives, which transform as the spinor of SO(9), as

$$\mathcal{Q}^a_+ = \partial_{\chi^a} + \frac{1}{\sqrt{2}}p^+\chi^a\,, \qquad \mathcal{Q}^a_- = -\frac{p^i}{p^+}\left(\gamma^i\mathcal{Q}_+\right)^a\,, \tag{3.4}$$

$$\begin{aligned} M^{ij} &= x^ip^j - x^jp^i - \frac{i}{2}\chi\,\gamma^{ij}\partial_\chi\,, \\ M^{+-} &= -x^-p^+ - \frac{i}{2}\chi\partial_\chi\,, \\ M^{+i} &= -x^ip^+\,, \\ M^{-i} &= x^-p^i - \frac{1}{2}\{x^i, P^-\} + \frac{ip^j}{2p^+}\chi\gamma^i\gamma^j\partial_\chi\,. \end{aligned} \tag{3.5}$$

The light-cone little group transformations are generated by

$$S^{ij} = -\frac{i}{2}\chi\gamma^{ij}\partial_\chi\,,$$

which satisfy the SO(9) Lie algebra. To construct its spectrum, we write the supercharges in terms of eight complex Grassmann variables

$$\theta^\alpha \equiv \frac{1}{\sqrt{2}}\left(\chi^\alpha + i\chi^{\alpha+8}\right)\,, \qquad \overline{\theta}^\alpha \equiv \frac{1}{\sqrt{2}}\left(\chi^\alpha - i\chi^{\alpha+8}\right)\,,$$

and

$$\frac{\partial}{\partial\theta^\alpha} \equiv \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\chi^\alpha} - i\frac{\partial}{\partial\chi^{\alpha+8}}\right)\,, \qquad \frac{\partial}{\partial\overline{\theta}^\alpha} \equiv \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\chi^\alpha} + i\frac{\partial}{\partial\chi^{\alpha+8}}\right)\,,$$

where $\alpha = 1, 2, \ldots, 8$. The eight complex θ transform as the $(\mathbf{4}\,,\,\mathbf{2})$, and $\overline{\theta}$ as the $(\overline{\mathbf{4}}\,,\,\mathbf{2})$ of the SU(4) $\times$ SU(2) subgroup of SO(9). The eight complex supercharges

$$\mathbf{Q}^\alpha_+ \equiv \frac{1}{\sqrt{2}}\left(\mathcal{Q}^\alpha_+ + i\,\mathcal{Q}^{\alpha+8}_+\right) = \frac{\partial}{\partial\overline{\theta}^\alpha} + \frac{1}{\sqrt{2}}p^+\theta^\alpha\,, \tag{3.6}$$

$$\mathbf{Q}^{\alpha\,\dagger}_+ \equiv \frac{1}{\sqrt{2}}\left(\mathcal{Q}^\alpha_+ - i\,\mathcal{Q}^{\alpha+8}_+\right) = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{\sqrt{2}}p^+\overline{\theta}^\alpha\,, \tag{3.7}$$

satisfy

$$\left\{\mathbf{Q}^\alpha_+, \mathbf{Q}^{\beta\,\dagger}_+\right\} = \sqrt{2}p^+\delta^{\alpha\beta}\,. \tag{3.8}$$

They act irreducibly on chiral superfields which are annihilated by the covariant derivatives

$$\left(\frac{\partial}{\partial\overline{\theta}^\alpha} - \frac{1}{\sqrt{2}}p^+\theta^\alpha\right)\,\Phi(y^-,\theta) = 0\,,$$

where

$$y^- = x^- - \frac{i\theta\overline{\theta}}{\sqrt{2}}\,.$$

Expansion of the superfield in powers of the eight complex θ's yields 256 components, with the following SU(4) $\times$ SU(2) properties

$$\begin{aligned}
1 &\sim (\mathbf{1},\mathbf{1})\,,\\
\theta &\sim (\mathbf{4},\mathbf{2})\,,\\
\theta\theta &\sim (\mathbf{6},\mathbf{3}) \oplus (\mathbf{10},\mathbf{1})\,,\\
\theta\theta\theta &\sim \left(\overline{\mathbf{20}},\mathbf{2}\right) \oplus \left(\overline{\mathbf{4}},\mathbf{4}\right)\,,\\
\theta\theta\theta\theta &\sim (\mathbf{15},\mathbf{3}) \oplus (\mathbf{1},\mathbf{5}) \oplus \left(\mathbf{20}',\mathbf{1}\right)\,,
\end{aligned}$$

and the higher powers yield the conjugate representations by duality. These make up the three SO(9) representations of $N = 1$ supergravity

$$\begin{aligned}
\mathbf{44} &= (\mathbf{1},\mathbf{5}) \oplus (\mathbf{6},\mathbf{3}) \oplus (\mathbf{20}',\mathbf{1}) \oplus (\mathbf{1},\mathbf{1})\,,\\
\mathbf{84} &= (\mathbf{15},\mathbf{3}) \oplus (\overline{\mathbf{10}},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1}) \oplus (\mathbf{6},\mathbf{3}) \oplus (\mathbf{1},\mathbf{1})\,,\\
\mathbf{128} &= (\mathbf{20},\mathbf{2}) \oplus (\overline{\mathbf{20}},\mathbf{2}) \oplus (\mathbf{4},\mathbf{4}) \oplus (\overline{\mathbf{4}},\mathbf{4}) \oplus (\mathbf{4},\mathbf{2}) \oplus (\overline{\mathbf{4}},\mathbf{2})\,.
\end{aligned}$$

For future reference we note the SU(4)$\times$SU(2) Dynkin weights of the θs, using the notation $(a_1, a_2, a_3;\, a)$,

$$\begin{aligned}
\theta^1 &\sim (1,0,0;1)\,, & \theta^8 &\sim (1,0,0;-1)\,,\\
\theta^4 &\sim (-1,1,0;1)\,, & \theta^5 &\sim (-1,1,0;-1)\,,\\
\theta^7 &\sim (0,-1,1;1)\,, & \theta^2 &\sim (0,-1,1;-1)\,,\\
\theta^6 &\sim (0,0,-1;1)\,, & \theta^3 &\sim (0,0,-1;-1)\,,
\end{aligned}$$

which enables us to find the highest weights of the supergravity representations

$$\begin{aligned}
\mathbf{44} : \theta^1\theta^4\theta^5\theta^8 &= (0,2,0;0) \sim (\mathbf{20'1}) \\
\mathbf{84} : \theta^1\theta^8 &= (2,0,0;0) \sim (\mathbf{10},\mathbf{1}) \\
\mathbf{128} : \theta^1\theta^4\theta^8 &= (1,1,0;1) \sim (\mathbf{20},\mathbf{2})\ ,
\end{aligned}$$

together with their $\mathrm{SU}(4)\times\mathrm{SU}(2)$ properties. All other states are generated by acting on these highest weight states with the lowering operators. The highest weight chiral superfield that describes $N=1$ supergravity in eleven dimensions is simply

$$\Phi = \theta^1\theta^8 h(y^-,\vec{x}) + \theta^1\theta^4\theta^8\psi(y^-,\vec{x}) + \theta^1\theta^4\theta^5\theta^8 A(y^-,\vec{x})\,, \tag{3.9}$$

which summarizes the spectrum of the super-Poincaré algebra in eleven dimensions of either a free field theory or a free superparticle. All other states are obtained by applying the SO(9) lowering operators.

Since the little group generators act on a 256-dimensional space, we can express them in terms of sixteen (256×256) matrices, Γ^a, which satisfy the Dirac algebra

$$\left\{\Gamma^a,\Gamma^b\right\} = 2\delta^{ab}\,. \tag{3.10}$$

This leads to an elegant representation of the SO(9) generators

$$S^{ij} = -\frac{i}{4}\left(\gamma^{ij}\right)^{ab}\Gamma^a\Gamma^b \ \equiv -\frac{i}{2}f^{ijab}\Gamma^a\Gamma^b\,. \tag{3.11}$$

The coefficients

$$f^{ijab} \equiv \frac{1}{2}\left(\gamma^{ij}\right)^{ab}\ ,$$

naturally appear in the commutator between the generators of SO(9) and any spinor operator T^a, as

$$\left[T^{ij},T^a\right] = \frac{i}{2}\left(\gamma^{ij}T\right)^a = if^{ijab}T^b\,. \tag{3.12}$$

But there is more to it, the $(\gamma^{ij})^{ab}$ can also be viewed as structure constants of a Lie algebra. Manifestly antisymmetric under $a\leftrightarrow b$, they can appear in the commutator of two spinors into the SO(9) generators

$$\left[T^a,T^b\right] = \frac{i}{2}\left(\gamma^{ij}\right)^{ab}T^{ij} = f^{abij}T^{ij}\,, \tag{3.13}$$

and one easily checks that they satisfy the Jacobi identities. Remarkably, the 52 operators T^{ij} and T^a generate the exceptional Lie algebra F_4, showing explicitly how an exceptional Lie algebra appears in the light-cone formulation of supergravity in eleven dimensions!

3.2 Character formula

As we have seen, the supergravity degrees of freedom are labeled by the light-cone little group SO(9) acting on the transverse plane indices, as $h_{(ij)}\sim(2000)$, $A_{[ijk]}\sim(0010)$, $\Psi_{i\,\alpha}\sim(1001)$, shown here in Dynkin's notation.

Their group-theoretical properties are summarized in the following table

irrep	(1001)	(2000)	(0010)
D	128	44	84
I_2	256	88	168
I_4	640	232	408
I_6	1792	712	1080
I_8	5248	2440	3000

where D is the dimension of the representation, and I_n are the Dynkin indices of the representations, related to the four Casimir operators of SO(9). We note that the dimension and Dynkin indices of the fermion is the sum over those of the bosons, except for I_8, indicating that these three representations have much in common. The failure of the sum rule for I_8 can be traced to the lack of renormalizability of the theory [2]. It is understood as the bosonic and fermionic representations stem from the two spinor representations of SO(16) which differ only in their highest order invariant.

It is a remarkable fact that the supergravity fields are the first of an infinite number of triplets [1] of SO(9) representations which display the same group-theoretical relations: equality of dimension and of all Dynkin indices except I_8 between one representation and the sum of the other two. Quantum theories of these Euler triplets may have very interesting divergence properties, as these numbers typically occur in higher loop calculations, and such equalities usually increase the degree of divergence, and the failure of the equality for I_8 is probably related to the lack of renormalizability of the theory [2].

This mathematical fact has been traced to a character formula [4] related to the three equivalent embeddings of SO(9) into F_4! This character formula is given by

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_c \mathrm{sgn}(c)\, \mathrm{U}_{c\bullet\lambda} \ .$$

On the left-hand side, V_λ is a representation of F_4 written in terms of its SO(9) subgroup, $S^\pm$ are the two spinor representations of SO(16) written in terms of its anomalously embedded subgroup SO(9), $\otimes$ denotes the normal Kronecker product of representations, and the $-$ denotes the naive substraction of representations. On the right-hand side, the sum is over c, the elements of the Weyl group which map the Weyl chamber of F_4 into that of SO(9). In this case there are three elements, the ratio of the orders of the Weyl groups (it is also the Euler number of the coset manifold), and $\mathrm{U}_{c\bullet\lambda}$ denotes the SO(9) representation with highest weight $c \bullet \lambda$, where

$$c \bullet \lambda = c(\lambda + \rho_{F_4}) - \rho_{\mathrm{SO}(9)} \,,$$

and the ρ's are the sum of the fundamental weights for each group, and sgn(c) is the index of c. Thus to each F_4 representation corresponds a triplet, called Euler triplet. The supergravity case is rather trivial as

$$\mathrm{SO}(16) \supset \mathrm{SO}(9)\,, \qquad S^+ \sim \mathbf{128} = \mathbf{128}\,, \qquad S^- \sim \mathbf{128'} = \mathbf{44} + \mathbf{84}\,,$$

and the character formula reduces to the truism

$$\mathbf{128} - \mathbf{44} - \mathbf{84} = \mathbf{128} - \mathbf{44} - \mathbf{84}\,.$$

This construction yields the general form of the Euler triplets: the Euler triplet corresponding to the F_4 representation $[\,a_1\,a_2\,a_3\,a_4\,]$ is made up of the following three SO(9) representations listed in order of increasing dimensions:

$$(2+a_2+a_3+a_4, a_1, a_2, a_3)\,, \quad (a_2, a_1, 1+a_2+a_3, a_4)\,, \quad (1+a_2+a_3, a_1, a_2, 1+a_3+a_4)$$

The spinor representations appear with odd entries in the fourth place. Euler triplets with the largest spinor and two bosons must have both a_3 and a_4 even or zero.

Since the Dynkin indices of the product of two representations satisfy the composition law

$$I^{(n)}[\lambda\otimes\mu] = d_\lambda I^{(n)}[\mu] + d_\mu I^{(n)}[\lambda]\,,$$

it follows that the deficit in $I^{(8)}$ is always proportional to

$$d_\lambda(I^{(8)}_{S+} - I^{(8)}_{S-})$$

where d_λ is the dimension of the F_4 representation that generates it. This yields the hope that by summing over an infinite set of representations of F_4, it might be possible to erase the deficit. Such a theory would presumably be finite.

3.3 The Kostant operator

This character formula can be viewed as the index formula of a Dirac-like operator [8] formed over the coset $F_4/\,\mathrm{SO}(9)$. This coset is the sixteen-dimensional Cayley projective plane, over which we introduce the previously considered Clifford algebra

$$\left\{\Gamma^a, \Gamma^b\right\} = 2\delta^{ab}, a, b = 1, 2, \ldots, 16\,, \tag{3.14}$$

generated by (256×256) matrices. The Kostant equation is defined as

$$\not{\mathcal{K}}\,\Psi = \sum_{a=1}^{16}\Gamma^a T^a\Psi = 0\,, \tag{3.15}$$

where T_a are F_4 generators not in SO(9), with commutation relations

$$\left[T^a, T^b\right] = i f^{abij} T^{ij}\,. \tag{3.16}$$

Although it is taken over a compact manifold, it has non-trivial solutions. To see this, we rewrite its square as the difference of positive definite quantities,

$$\not{\mathcal{K}}\not{\mathcal{K}} = C^2_{F_4} - C^2_{\mathrm{SO}(9)} + 72\,, \tag{3.17}$$

where

$$C^2_{F_4} = \frac{1}{2}T^{ij}T^{ij} + T^aT^a\,,$$

is the F_4 quadratic Casimir operator, and

$$C^2_{\mathrm{SO}(9)} = \frac{1}{2}\left(T^{ij} - if^{abij}\widetilde{\Gamma}^{ab}\right)^2 ,$$

is the quadratic Casimir for the sum

$$L^{ij} \equiv T^{ij} + S^{ij} ,$$

where S^{ij} is the previously defined SO(9) generator which acts on the supergravity fileds. We have also used the quadratic Casimir on the spinor representation

$$\frac{1}{2}S^{ij}S^{ij} = 72 .$$

Kostant's operator commutes with the sum of the generators,

$$[\not{K}, L^{ij}] = 0 ,$$

allowing its solutions to be labelled by SO(9) quantum numbers.

The same construction of Kostant's operator applies to all equal rank embeddings, and its trivial solutions display supersymmetry [4, 5, 6, 7]. In particular we note the cases $E_6/\,\mathrm{SO}(10) \times \mathrm{SO}(2)$, with Euler number 27, $E_7/\,\mathrm{SO}(12) \times \mathrm{SO}(3)$ with Euler number 63, and $E_8/\,\mathrm{SO}(16)$, where the Euler triplets contain 135 representations [1]. These cosets with dimensions $32, 64$, and 128 could be viewed as complex, quaternionic and octonionic Cayley plane [13].

3.4 Solutions of Kostant's equation

For every representation of F_4, $[a_1, a_2, a_3, a_4]$, there is one Euler triplet solution of Kostant's equation containing the three SO(9) representations

$$(2 + a_2 + a_3 + a_4, a_1, a_2, a_3) , \quad (a_2, a_1, 1 + a_2 + a_3, a_4) , \quad (1 + a_2 + a_3, a_1, a_2, 1 + a_3 + a_4) .$$

The trivial solution with $a_1 = a_2 = a_3 = a_4 = 0$, yields the $N = 1$ supergravity multiplet in eleven dimensions, $(2000) \oplus (0010) \oplus (1001)$. In our representation, the highest weight solution are $\theta^1\theta^4\theta^5\theta^8, \theta^1\theta^8$, and $\theta^1\theta^4\theta^8$, described by the chiral superfield

$$\Phi_{0000} = \theta^1\theta^8 h(y^-, \vec{x}) + \theta^1\theta^4\theta^8\, \psi(y^-, \vec{x}) + \theta^1\theta^4\theta^5\theta^8 A(y^-, \vec{x}) . \tag{3.18}$$

The rest of the solutions contain "internal" variables on which F_4 acts, in exact analogy with the z_i of the previous section. A particularly convenient Schwinger-like representation F_4 can be found in Appendix C. It introduces three sets of "internal" bosonic coordinates, u_i which transform as an SO(9) vector, and ζ_a which transform as an SO(9) spinor. Curiously, an SO(9)-scalar coordinate u_0 required in the construction of the generators does not appear in the solutions of Kostant's equation. These coordinates span the fundamental **26** representation of F_4. Note the appearance of a bosonic coordinates with spinorial properties, in apparent violation of the spin-statistics connection.

In general, the highest weight solutions appear in the form of $f(u_i,\zeta_a)\,\Theta(\theta)$, where both $f(u_i,\zeta_a)$ and $\Theta(\theta)$ are the highest weights SO(9) states with respect to T_{ij} and S_{ij}. The solutions have the quantum numbers of the algebra generated by their sum $L_{ij} = S_{ij} + T_{ij}$, which commutes with Kostant's operator. $\Theta(\theta)$ is one of the three polynomials, $\theta^1\theta^4\theta^5\theta^8$,$\theta^1\theta^8$, or $\theta^1\theta^4\theta^8$.

It suffices to find the highest weight solutions, as all other solutions are obtained by the action of the four lowering SO(9) operators. The highest weight solutions corresponding to each fundamental representation of F_4 are

1. $a_1 = a_2 = a_3 = 0$, $a_4 \geq 1$

   These representations require only one copy of the internal coordinates. The highest weight solutions with $a_4 = 1$ are given by

$$\theta^1\theta^4\theta^5\theta^8 w_4\,, \qquad \theta^1\theta^8 v_4\,, \qquad \theta^1\theta^4\theta^8 v_4\,, \tag{3.19}$$

   where

$$w_4 = (u_1 + iu_2)\,, \qquad v_4 = (\zeta_1 + i\zeta_9)\,, \tag{3.20}$$

   are the highest weights of SO(9) representations (1000) and (0001), respectively.

2. $a_1 = a_2 = a_4 = 0$, $a_3 \geq 1$

   This case demands two copies. For $a_3 = 1$, the highest weight solutions are

$$\theta^1\theta^4\theta^5\theta^8 w_3\,, \qquad \theta^1\theta^8 v_3\,, \qquad \theta^1\theta^4\theta^8 w_3\,, \tag{3.21}$$

   where

$$w_3 = [u_1 + iu_2, \zeta_1 + i\zeta_9]\,, \qquad v_3 = [\zeta_1 + i\zeta_9, \zeta_8 + i\zeta_{16}]\,, \tag{3.22}$$

   are the highest weights of the SO(9) representations (1001) and (0010) respectively, and

$$[a,b] \equiv a^{[1]}b^{[2]} - a^{[2]}b^{[1]}\,,$$

   stands for the determinant of 2 copies of a and b states.

3. $a_1 = a_3 = a_4 = 0$, $a_2 \geq 1$

   Here three copies are needed. The $a_2 = 1$ highest weight solutions are

$$\theta^1\theta^4\theta^5\theta^8 w_2\,, \qquad \theta^1\theta^8 w_2\,, \qquad \theta^1\theta^4\theta^8 w_2\,, \tag{3.23}$$

   where

$$w_2 = [u_1 + iu_2, \zeta_1 + i\zeta_9, \zeta_8 + i\zeta_{16}]\,, \tag{3.24}$$

   is the highest weight of the SO(9) representation (1010), and $[\,a\,,\,b\,,\,c\,]$ is the determinant (antisymmetric product) of 3 copies of a, b and c states. The same combination of internal variables appears for all three highest weight components.

4. $a_2 = a_3 = a_4 = 0$, $a_1 = 1$

   The F_4 states are represented by antisymmetric products of two copies of 26 states. The highest weight solutions are

$$\theta^1\theta^4\theta^5\theta^8 w_1\,, \qquad \theta^1\theta^8 w_1\,, \qquad \theta^1\theta^4\theta^8 w_1\,, \tag{3.25}$$

   where

$$w_1 = [u_1 + iu_2, u_3 + iu_4] + [\zeta_1 + i\zeta_9, \zeta_6 - i\zeta_{14}] + [\zeta_8 + i\zeta_{16}, \zeta_3 - i\zeta_{11}]\,, \tag{3.26}$$

   is the highest weight of the SO(9) representation (0100). This is the algebraically most complicated case, but all three members of the triplet appear multiplied by the same combination of internal variables.

This last case implies that only three copies of 26 oscillators suffice to generate all F_4 representations. It is not possible to construct the $[\,1\,0\,0\,0\,]$ state out of four copies of states in the **26**, and all representations of F_4 can be obtained by three copies of harmonic oscillator variables.

Since the T_{ij} do not alter the degree of homogeneity of polynomials in u_i and ζ_a, all solutions are given by the functions $f(u_i, \zeta_a)$ as homogeneous polynomials of their variables. The general highest weight solutions are then

$$\begin{aligned}
&\theta^1\theta^4\theta^5\theta^8\, w_1^{a_1}\, w_2^{a_2}\, w_3^{a_3}\, w_4^{a_4}\,,\\
&\theta^1\theta^8\, w_1^{a_1}\, w_2^{a_2}\, v_3^{a_3}\, v_4^{a_4}\,,\\
&\theta^1\theta^4\theta^8\, w_1^{a_1}\, w_2^{a_2}\, w_3^{a_3}\, v_4^{a_4}\,.
\end{aligned}$$

All other states are generated by application of the four SO(9) lowering operators. These explicit forms of the solutions to Kostant's equation, as products of a θ part and an internal part that depends polynomially on new variables, may suggest a physical interpretation. Indeed, as the θ parts describe a superparticle in eleven dimensions, it is tempting to interpret states in the other Euler triplets as superparticles dressed with fields described by these new variables, vector coordinates $u_i^{[\kappa]}$ and twistor-like coordinates $\zeta_a^{[\kappa]}$.

3.5 Super Euler triplets

Any solution with odd powers ζ_a describes fermions (SO(9) spinors) with Bose properties, in contradiction with the spin-statistics connection. To avoid this conflict, we must restrict our attention to Euler triplets with even powers of ζ_a; from the explicit forms of the solutions, this means Euler triplets for which a_3 and a_4 are even, with no restrictions on a_1 and a_2.

The supergravity Euler triplet contains equal number of fermions and bosons. Although none of the other Euler triplets display space-time supersymmetry, we may ask which subclass contains equal number of fermions (SO(9) spinors) and bosons (SO(9) tensors). Curiously, these are the same triplets as those required by spin-statistics, those for which a_3 and a_4 are even.

In this section we concentrate on these special triplets which we call Super Euler triplets (SETs) that share this one feature of supersymmetry, although they are not space-time supersymmetric by themselves. There are four different SET families:

- The simplest has a_4 even, and $a_3 = a_2 = a_1 = 0$. The general highest weight superfield in this class is given by (with generic spacetime fields)

$$\Phi_4^n = \phi(y^-, \vec{x})\theta^1\theta^4\theta^5\theta^8 w_4^{2n} + A(y^-, \vec{x})\theta^1\theta^8 v_4^{2n} + \psi(y^-, \vec{x})\theta^1\theta^4\theta^8 v_4^{2n} \,. \tag{3.27}$$

  This family requires one new set of vector and twistor coordinates. For $n = 1$ the internal coordinates generate a symmetric second rank tensor represented by (2000) coupled to gravity, and a four-form (0002), coupled to the three-form and Rarita-Schwinger fields of supergravity.

- The case $a_3 \neq 0$ even, and $a_1 = a_2 = a_4 = 0$ requires two sets of extra coordinates, as its highest weight superfield is

$$\Phi_3^n = \phi(y^-, \vec{x})\theta^1\theta^4\theta^5\theta^8 w_3^{2n} + A(y^-, \vec{x})\theta^1\theta^8 v_3^{2n} + \psi(y^-, \vec{x})\theta^1\theta^4\theta^8 w_3^{2n} \,. \tag{3.28}$$

  The two sets of internal coordinates arrange themselves in the symmetric product of two three forms that couple to the supergravity three-form, and two RS spinors that couples to gravity and the original RS fields of supergravity.

- When only $a_2 \neq 0$, all three supergravity multiplets are dressed by the same (1010) representation, described by triple products of vector and two spinors. The highest weight superfield is now

$$\Phi_2^n = \left[\phi(y^-, \vec{x})\theta^1\theta^4\theta^5\theta^8 + A(y^-, \vec{x})\theta^1\theta^8 + \psi(y^-, \vec{x})\theta^1\theta^4\theta^8\right] w_2^n \,, \tag{3.29}$$

  which requires three sets of coordinates.

- Finally, if $a_1 \neq 0$ only, the three supergravity states are dressed the same way by a combination of two vectors and two spinors with the quantum number of a 2-form, (0100). Although it is the most complicated in terms of the underlying coordinates, it is the simplest in terms of representations.

$$\Phi_1^n = \left[\phi(y^-, \vec{x})\,\theta^1\theta^4\theta^5\theta^8 + A(y^-, \vec{x})\theta^1\theta^8 + \psi(y^-, \vec{x})\theta^1\theta^4\theta^8\right] w_4^n \,. \tag{3.30}$$

  These SETs require only two copies. This doubling may indicate the presence of E_6, the complex extension of F_4.

We note in passing that none of the solutions depend on the singlet variable u_0, which can be traced to the equation

$$\Gamma^a \frac{\partial}{\partial \zeta_a} = 0 \,.$$

We already know that space-time supersymmetry cannot be implemented on any one of these triplets, although they may still be relativistic. For these Euler triplets to describe relativistic states, one must be able to implement their Poincaré transformations. In the light-cone form, the kinematic generators are already known, since the light-cone little group is generated by the sum $L^{ij} = T^{ij} + S^{ij}$, where S^{ij} acts on the Grassmann variables and T^{ij} on the internal variables. In order to determine the dynamic generators, one needs to find the mass operator. One of two possibilities arise

- The Euler triplet states are massless, and there is no further addition to the light-cone hamiltonian P^-. These states represent massless higher spin particles, with well-known difficulties in implementing their interactions [9]. In this case an interacting theory would need to use an infinite number of Euler triplets to avoid these no-go theorems [10].

- If the excited Euler triplets describe massive states, a non-linear realization of the generators of the massive little group SO(10) with the SO(9) part given by L^{ij} must exist. This requires the construction of a transverse vector L^i, with the commutation relations
$$[L^i, L^j] = iM^2 L^{ij} ,$$
where M^2 is the mass squared operator which commutes with L^{ij}. If such an operator can be found, one easily builds the light-cone boosts which satisfy the Poincaré algebra. For strings and superstrings, the L^i are cubic in the oscillators, and this commutation relation works only in the right number of dimensions.

It is not possible to find such a non-linear representation of SO(10) using only the degrees of freedom present in the Super Euler triplets, so that their relativistic description requires them to be massless. To see this, consider the SO(9) fermionic representation of the $\{a_1\, a_2\, a_3\, a_4\}$ Super Euler triplet, with Dynkin labels $(1 + a_2 + a_3, a_1, a_2, 1 + a_3 + a_4)$ and a_3, a_4 even. It can be realized in terms of fields with one spinor index and a tensor structure given by the partition
$$\left[1 + a_1 + a_2 + a_3 + \frac{a_3 + a_4}{2}, a_1 + a_2 + \frac{a_3 + a_4}{2}, a_2 + \frac{a_3 + a_4}{2}, \frac{a_3 + a_4}{2}\right] ,$$
which has the first row of its Young tableau always larger than its second row. This representation is contained within an SO(10) spinor-tensor which also has more in its first row than in its second. However such a representation, expressed in SO(9) language, also contains a partition in which the excess in the first row is identified with the tenth direction, resulting in an SO(9) spinor-tensor with equal numbers in its first two rows. Since this representation is not in any super triplet, we must conclude that it is not possible to build massive relativistic Super Euler triplets without introducing new degrees of freedom.

At this stage, our approach still lacks an organizing principle to determine which set of Euler triplets to include. A likely first restriction, from the spin-statistics connection, limits the multiplets to those with (a_3, a_4 even). A second one is to require that the deficit in $I^{(8)}$ be erased by summing over an infinite set of Euler triplets. For instance an infinite sum of F_4 dimensions involves a 24th order polynomial over the Dynkin indices. Since it is an even power, it could vanish as $\zeta(-2m) = 0$. We hope to come back to this point in a future publication. This also suggests the inclusion of F_4 inside a non-compact structure.

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We see that the Euler triplets generate the spectrum of a Poincaré covariant object which has a ground state with supersymmetry! This object would be described by its center of mass coordinates x^- and x_i, and internal coordinates $u_i^{(\kappa)}, \zeta_a^{(\kappa)}$, where κ may run over three values at most. Contrast this with the superstring which is also described by its center of mass coordinates x^-, x_m, $m = 1, \ldots, 8$ and an infinite number of internal variables $x_m^{(n)}$, and anticommuting spinor variables $\zeta_\alpha^{(n)}$, $n = 1, 2, \ldots, \infty$, with $\alpha = 1, \ldots, 8$.

In this language, the Euler triplets emerge as much simpler than superstrings, since they have a finite number of internal variables, although their internal spinor variables satisfy Bose commutations.

Could these label the end points of an open string in the zero tension limit?

Could this new internal space be generated by the degrees of freedom of the Exceptional Jordan Algebras?

4. The action for a Kostant supermultiplet

In the previous sections we have described the free system of states which are subject to Kostant's equation, using the light-cone formalism. The action that describes these states as a set of point-like particles is gauge-fixed. It would be very interesting to find the gauge-invariant action, but we have so far failed to do so. Even the gauge-fixed action is interesting as it requires Kostant's equation as a gauge condition.

We write this action in the light cone frame, in a generic form to cover the case we have analyzed. The superspace dynamical coordinates will be x^i with $i = 1, 2, \ldots d-2$ and the spinorial ones S^a with $a = 1, \ldots 16$ for $d = 11$ and correspondingly for other values of d. The internal ones will be z^A and $\overline{z}_A$, where $A = 1, 2, 3$ for the case of $\mathrm{SU}(3)/\mathrm{SU}(2) \times \mathrm{U}(1)$ and $A = A_n$, where $A = 1, \ldots 26$ and $n = 1, \ldots 3$ (or 4) for the case of $F_4/\mathrm{SO}(9)$.

We then can write the following action

$$S = \int d\tau \left(\frac{1}{2} \left(\dot{x}^i\right)^2 + iS^a \dot{S}^a + \dot{z}^A \dot{\overline{z}}_A + \lambda S^a T^a \right) , \tag{4.1}$$

where T^a are the generators for the quotient space (see the appendices), where we have exchanged $\partial/\partial z$ with $i\dot{\overline{z}}$ and the same for the complex conjugate

All coordinates are functions of τ. The Lagrange multiplier λ gives the Kostant constraint as an equation of motion. In Dirac's language the constraint arises since the momentum conjugate to λ, π_λ is 0. This is a constraint on the phase space and signals a gauge invariance. A typical gauge choice that we will make in the sequel at the level of the equations of motion is $\lambda = 0$. It seems difficult to fix the gauge and put it back to the action as we have done with the local symmetries that allowed us to just use x^i and S^a as dynamical coordinates. We have fixed a local reparametrization symmetry to eliminate x^- and p^-. We believe that some kind of κ-symmetry has allowed us to eliminate half a spinor. Presumably another bosonic symmetry has allowed us to eliminate z-variables down to the ones above in the action. In the gauge where $\lambda = 0$ the equations of motion are

$$\ddot{x}^i = 0 \,, \qquad \dot{S}^a = 0 \,, \tag{4.2}$$

$$\ddot{z}^a = 0\,, \qquad \ddot{\overline{z}}_A = 0\,, \tag{4.3}$$

$$S^a T^a = 0 \tag{4.4}$$

The canonical commutation relations are

$$\left[x^i, p^j\right] = i\delta^{ij}\,, \qquad \left\{S^a, S^b\right\} = \delta^{ab} \tag{4.5}$$

$$\left[z^A, p_{z\,B}\right] = i\delta^A{}_B\,, \qquad \left[\overline{z}_A, \overline{p}_z{}^B\right] = i\delta_A{}^B\,, \tag{4.6}$$

where the conjugate momenta are

$$p^i = \dot{x}^i\,, \qquad p_{z\,A} = \dot{\overline{z}}_A\,, \qquad \overline{p}_z{}^A = \dot{z}^A\,. \tag{4.7}$$

If we insert the p_z for $\dot{z}$ in T^a we get the form of the qoutient space generators.

We can represent the S^a's in two different ways. In the first case we just choose

$$S^a = \frac{1}{\sqrt{2}}\gamma^a\,. \tag{4.8}$$

The second case we illustrate for the case when $a = 1, 2$. We choose

$$S^1 + iS^2 = \frac{\partial}{\partial\overline{\theta}} + \theta \tag{4.9}$$

and

$$S^1 - iS^2 = \frac{\partial}{\partial\theta} + \overline{\theta} \tag{4.10}$$

We have now introduced too many degrees of freedom so we have to impose the chiral constraint. We have put $p^+ = 1$ in all these arguments to streamline it. We can then represent our particle on wave functions which are functions of a commuting set of the phase space coordinates, ie. x^+, p^+, x^i, θ, $\overline{\theta}$, z^A and $\overline{z}_A$, which are chiral and are subject to the Kostant constraint.

We notice that the two constraints, the chiral one and the Kostant one come from very different causes. The chiral condition comes from using a too big space while the Kostant constraint comes from a gauge invariance.

The action is invariant under the full Poincare algebra where the SO(9) part includes the z-dependence. The kinetic term for the z's has the full F_4 invariance, but this symmetry is broken by the Kostant term.

Suppose we now try to introduce back the full gauge invariance. We can introduce a reparametrization invariance by using an einbein, but if we introduce a κ-invariance this will affect the Kostant term too, since we then have to double the spinor. What happens then to the Kostant term? Is there a group with a quotient space which is 32-component. We know of the complex Cayley plane $E_6/\,\mathrm{SO}(10) \times \mathrm{SO}(2)$, but we have so far not been able to complete this argument.

A. SU(3) oscillator representations

Schwinger's celebrated representation of SU(2) generators [14] in terms of one doublet of harmonic oscillators has been extended to other Lie algebras [15]. The generalization involves several sets of harmonic oscillators, each spanning the fundamental representations. Thus SU(3) is generated by two sets of triplet harmonic oscillators, one transforming as a triplet the other as an antitriplet. Its generators are given by

$$\begin{aligned} T_1 + iT_2 &= z_1\partial_2 - \overline{z}_2\overline{\partial}_1\,, \qquad & T_1 - iT_2 &= z_2\partial_1 - \overline{z}_1\overline{\partial}_2\,, \\ T_4 + iT_5 &= z_1\partial_3 - \overline{z}_3\overline{\partial}_1\,, \qquad & T_4 - iT_5 &= z_3\partial_1 - \overline{z}_1\overline{\partial}_3\,, \\ T_6 + iT_7 &= z_2\partial_3 - \overline{z}_3\overline{\partial}_2\,, \qquad & T_6 - iT_7 &= z_3\partial_2 - \overline{z}_2\overline{\partial}_3\,, \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} T_3 &= \frac{1}{2}\left(z_1\partial_1 - z_2\partial_2 - \overline{z}_1\overline{\partial}_1 + \overline{z}_2\overline{\partial}_2\right)\,, \\ T_8 &= \frac{1}{2\sqrt{3}}\left(z_1\partial_1 + z_2\partial_2 - \overline{z}_1\overline{\partial}_1 - \overline{z}_2\overline{\partial}_2 - 2z_3\partial_3 + 2\overline{z}_3\overline{\partial}_3\right)\,, \end{aligned} \tag{A.2}$$

where we have defined

$$\partial_1 \equiv \frac{\partial}{\partial z_1}\,, \qquad \overline{\partial}_1 \equiv \frac{\partial}{\partial \overline{z}_1}\,, \qquad \text{etc.}\,.$$

These act as hermitean operators on holomorphic functions of $z_{1,2,3}$ and $\overline{z}_{1,2,3}$, normalized with respect to the inner product

$$(f,g) \equiv \int d^3z d^3\overline{z} e^{-\sum_i |z_i|^2} f^*(z,\overline{z}) g(z,\overline{z})\,.$$

Acting on the highest-weight states, the SU(3) quadratic Casimir operator is

$$C_2^{\mathrm{SU}(3)} \equiv \sum_{a=1}^{8} T_a T_a \Big|_{\text{highest weight}} = T_3(T_3+1) + T_8(T_8+\sqrt{3})\,.$$

The second Casimir operator is cubic, and of no concern here. Rather than labelling the representations in terms of their eigenvalues, it is more convenient to introduce the positive integer Dynkin labels a_1 and a_2. We have

$$T_3|a_1,a_2\rangle = \frac{a_1}{2}|a_1,a_2\rangle\,, \qquad T_8|a_1,a_2\rangle = \frac{1}{2\sqrt{3}}(a_1+2a_2)|a_1,a_2\rangle\,,$$

so that

$$C_2^{\mathrm{SU}(3)} = (a_1+a_2) + \frac{1}{3}(a_1^2 + a_1a_2 + a_2^2)\,.$$

The highest-weight states of each SU(3) representation are holomorphic polynomials of the form

$$z_1^{a_1}\overline{z}_3^{a_2}\,,$$

where a_1, a_2 are its Dynkin indices: all representations of SU(3) are homogeneous holomorphic polynomials.

Finally we note that the Casimir operator of the SU(2) subalgebra is given by

$$\vec{T}\cdot\vec{T} = \frac{1}{4}D_\perp(D_\perp + 2)\,, \tag{A.3}$$

where

$$D_\perp = z_1\partial_1 + z_2\partial_2 + \overline{z}_1\overline{\partial}_1 + \overline{z}_2\overline{\partial}_2\,,$$

so that the spin of the SU(2) representation is simply

$$J = \frac{1}{2}D_\perp\,. \tag{A.4}$$

B. Euler triplet ladder operations

In section 2 we treated the highest weight states and showed how to go from one state to another. In this Appendix we will treat the general case. Start with the general form of the Euler triplet superfields

$$\begin{aligned}\Phi_{\{a_1,a_2\}}(y^-,x^i;\mathbf{z},\overline{\mathbf{z}},\theta_m) \;=\; & \psi^0_{\{a_1,a_2\}}(y^-,x^i;\mathbf{z},\overline{\mathbf{z}}) + \theta_m\psi^m_{\{a_1,a_2\}}(y^-,x^i;\mathbf{z},\overline{\mathbf{z}}) + \\ & + \theta_1\theta_2\psi^{12}_{\{a_1,a_2\}}(y^-,x^i;\mathbf{z},\overline{\mathbf{z}})\,,\end{aligned} \tag{B.1}$$

where the components are homogeneous polynomials or order a_1 in the variables $\mathbf{z}$ and a_2 in the $\overline{\mathbf{z}}$. The simplest examples are

- The "trivial" $\{0,0\}$ Euler triplet, described by the superfield

  $$\Phi_{\{0,0\}}(y^-,x^i,\theta_m) = \psi^0_{\{0,0\}}(y^-,x^i) + \theta_m\psi^m_{\{0,0\}}(y^-,x^i) + \theta_1\theta_2\psi^{12}_{\{0,0\}}(y^-,x^i)\,, \tag{B.2}$$

  where the ψ's depend only on the center of mass variables y^-, and x_i, cf. (2.8), not on the internal z variables. The kinematic $N=2$ supersymmetry acts on its components by means of the operators

  $$\mathcal{Q}^m_+ \approx -2p^+\theta_m\,, \qquad \overline{\mathcal{Q}}^m_+ \approx \frac{\partial}{\partial\theta_m}\,, \tag{B.3}$$

  so that this superfield describes the $N=2$ Hypermultiplet on the light-cone.

- The $\{1,0\}$ triplet components are linear polynomials in the z_i

  $$\begin{aligned}\psi^0_{\{1,0\}} &= b_0z_3\,, & \psi^{12}_{\{1,0\}} &= b_{12}z_1 + b'_{12}z_2\,,\\ \psi^1_{\{1,0\}} &= b_1z_1 + b_2z_2\,, & \psi^2_{\{1,0\}} &= b_2z_1 + b_3z_2\,,\end{aligned} \tag{B.4}$$

  where the b's depend only on the center of mass variables.

- Similarly, the $\{2,0\}$ Euler triplet is represented by the quadratic polynomials

  $$\begin{aligned}\psi^0_{\{2,0\}} &= c_0z_3^2\,, & \psi^{12}_{\{2,0\}} &= c_{12}z_1^2 + c'_{12}z_1z_2 + c''_{12}z_2^2\,,\\ \psi^1_{\{2,0\}} &= c_1z_1^2 + 2c_2z_1z_2 + c_3z_2^2\,, & \psi^2_{\{2,0\}} &= c_2z_1^2 + 2c_3z_1z_2 + c_4z_2^2\,,\end{aligned} \tag{B.5}$$

  with the c's depending on the center of mass variables as above. Note that while ψ^0 and ψ^{12} are arbitrary polynomials of a given degree of homogeneity, some of the same coefficients appear in ψ^1 and ψ^2.

If we want to generate *all* states of an Euler triplet, then we need to consider complicated projection schemes that single out the requisite states. As an example, let us see how the $\{1,0\}$ and $\{2,0\}$ triplets are generated from the supersymmetric $\{0,0\}$. When applied to the lowest Euler triplet, the SU(2)-doublet operator $z_i\,(1-P_0)$, where $(1-P_0)$ is the projection operator onto the θ-dependent terms, generates more than the states in $\Psi_{\{1,0\}}$. In order to single out the requisite combinations of z's and θ's, we need the operator

$$\mathcal{P} = \frac{1}{2}\left(1 + \sum_{i,j=1,2} z_i\theta_j \frac{\partial}{\partial z_j}\frac{\partial}{\partial \theta_i}\right). \tag{B.6}$$

It carries no helicity and is an SU(2) singlet, as we can see by rewriting it in the form

$$\mathcal{P} = \frac{1}{2}\left(1 + 2\vec{S}\cdot\vec{T} + \frac{1}{2}\theta_i\frac{\partial}{\partial\theta_i}\left(z_j\frac{\partial}{\partial z_j} + \overline{z}_j\frac{\partial}{\partial\overline{z}_j}\right)\right). \tag{B.7}$$

Acting on the terms linear in θ's, it reduces to

$$\mathcal{P} = \frac{1}{2} + \vec{S}\cdot\vec{T} + J/2\,, \tag{B.8}$$

while it is equal to one-half acting on the $1\,,\ \theta_1\theta_2$ terms. With the help of

$$\left(\vec{S}\cdot\vec{T}\right)^2 = -\frac{1}{2}\vec{S}\cdot\vec{T} + \frac{1}{3}\vec{S}\cdot\vec{S}J(J+1)\,, \tag{B.9}$$

we see that on the terms linear in θ's,

$$\mathcal{P}^2 = \left(J + \frac{1}{2}\right)\mathcal{P}\,. \tag{B.10}$$

so that

$$\mathcal{M} = \frac{\mathcal{P}}{J+\frac{1}{2}}\,, \tag{B.11}$$

is a true projection operator on the θ-linear terms. It is easy to check that

$$\begin{gathered}\mathcal{P}z_1\theta_1 = z_1\theta_1\,; \qquad \mathcal{P}z_2\theta_2 = z_2\theta_2\,;\\ \mathcal{P}(z_2\theta_1 + z_1\theta_2) = (z_2\theta_1 + z_1\theta_2)\,,\end{gathered} \tag{B.12}$$

while the unwanted combination

$$\mathcal{P}\ (z_2\theta_1 - z_1\theta_2) = 0\,, \tag{B.13}$$

is annihilated. Multiplying the θ-dependent superfield components in the SU(2) spin j representation by z_i, produces two SU(2) representations, $j+1/2$ and $j-1/2$. The action of $\mathcal{P}$ is to project out the latter (like the well-known projection that appears in $L-S$ coupling problems). Hence the SU(2)-doublet “creation” operators have the simple form

$$\mathcal{A}_i^\dagger \equiv \mathcal{P}(1-P_0)z_i\,. \tag{B.14}$$

Explicit computations yield

$$\begin{aligned}
\mathcal{A}_0^\dagger \Psi_{\{0,0\}} &= \psi^0_{\{0,0\}} z_3 \\
\mathcal{A}_1^\dagger \Psi_{\{0,0\}} &= \psi^1_{\{0,0\}} z_1\theta_1 + \frac{1}{2}\psi^2_{\{0,0\}}(z_1\theta_2 + z_2\theta_1) + \psi^{12}_{\{0,0\}} z_1\theta_1\theta_2\,, \\
\mathcal{A}_2^\dagger \Psi_{\{0,0\}} &= \frac{1}{2}\psi^1_{\{0,0\}}(z_1\theta_2 + z_2\theta_1) + \psi^2_{\{0,0\}} z_2\theta_2 + \psi^{12}_{\{0,0\}} z_2\theta_1\theta_2\,,
\end{aligned}$$

so that their action on the lowest Euler triplet generates all the states in $\{1,0\}$, although one of the states is generated in two ways. The same redundancy is seen in the double application of the step-up operators on the lowest Euler triplet,

$$\begin{aligned}
(\mathcal{A}_0^\dagger)^2 \Psi_{\{0,0\}} &= \psi^0_{\{0,0\}} z_3^2\,, \\
(\mathcal{A}_1^\dagger)^2 \Psi_{\{0,0\}} &= \frac{3}{2}\psi^1_{\{0,0\}} z_1^2\theta_1 + \psi^2_{\{0,0\}}(z_1^2\theta_2 + 2z_1z_2\theta_1) + \frac{3}{2}\psi^{12}_{\{0,0\}} z_1^2\theta_1\theta_2\,, \\
(\mathcal{A}_2^\dagger)^2 \Psi_{\{0,0\}} &= \psi^1_{\{0,0\}}(z_2^2\theta_1 + 2z_1z_2\theta_2) + \frac{3}{2}\psi^2_{\{0,0\}} z_2^2\theta_2 + \frac{3}{2}\psi^{12}_{\{0,0\}} z_2^2\theta_1\theta_2\,, \\
\mathcal{A}_1^\dagger\mathcal{A}_2^\dagger \Psi_{\{0,0\}} &= \frac{1}{2}\psi^2_{\{0,0\}}(z_2^2\theta_1 + 2z_1z_2\theta_2) + \psi^1_{\{0,0\}}(z_1^2\theta_2 + 2z_1z_2\theta_1) + \frac{3}{2}\psi^{12}_{\{0,0\}} z_1z_2\theta_1\theta_2\,, \\
\mathcal{A}_2^\dagger\mathcal{A}_1^\dagger \Psi_{\{0,0\}} &= \frac{1}{2}\psi^1_{\{0,0\}}(z_1^2\theta_2 + 2z_1z_2\theta_1) + \psi^2_{\{0,0\}}(z_2^2\theta_1 + 2z_1z_2\theta_2) + \frac{3}{2}\psi^{12}_{\{0,0\}} z_1z_2\theta_1\theta_2\,.
\end{aligned}$$

Only the states in the $\{2,0\}$ multiplet are generated, although the same states can be generated by different sets of operators. This construction generalizes easily to all triplets of the form $\{a_1,0\}$: acting on any triplet $\Psi_{\{a_1,0\}}$, the step-up operators $\mathcal{A}^\dagger_{0,1,2}$ yield all the states in the three SU(2) representations of the triplet $\{a_1+1,0\}$. A similar construction holds for the step-down operators.

It is also easy to construct the ladder operators that relate triplets of the form $\Psi_{\{0,a_2\}}$. These are defined as

$$\overline{\mathcal{A}}^\dagger_{12} \equiv P_{12}\bar{z}_3\,, \qquad \overline{\mathcal{A}}^\dagger_i \equiv \bar{\mathcal{P}}(1-P_{12})\bar{z}_i\,, \tag{B.15}$$

where

$$\bar{\mathcal{P}} = \left(1 - \epsilon_{ij}\epsilon_{kl}\bar{z}_i\theta_k \frac{\partial}{\partial\bar{z}_l}\frac{\partial}{\partial\theta_j}\right). \tag{B.16}$$

and the corresponding "annihilation"operators

$$\overline{\mathcal{A}}_{12} \equiv P_{12}\frac{\partial}{\partial\bar{z}_3}\,, \qquad \overline{\mathcal{A}}_i \equiv \bar{\mathcal{P}}(1-P_{12})\frac{\partial}{\partial\bar{z}_i}\,. \tag{B.17}$$

We can define the operators that generate shadow supersymmetries in terms of these new ladder operators

$$\mathcal{Q}^{[i]}_{ab} \equiv \mathcal{A}^\dagger_a\theta_i\mathcal{A}_b\,, \tag{B.18}$$

where $a,b = 0,1,2$, and $i = 1,2$, and their inverses

$$\overline{\mathcal{Q}}^{[i]}_{ab} \equiv \mathcal{A}^\dagger_a\frac{\partial}{\partial\theta_i}\mathcal{A}_b\,. \tag{B.19}$$

Similarly we define the operators

$$Q_{ab}^{[i]} \equiv \bar{\mathcal{A}}_a^\dagger \theta_i \bar{\mathcal{A}}_b\,, \tag{B.20}$$

where $a, b = 0, 1, 2$, and $i = 1, 2$, and their inverses

$$\overline{Q}_{ab}^{[i]} \equiv \bar{\mathcal{A}}_a^\dagger \frac{\partial}{\partial \theta_i} \bar{\mathcal{A}}_b\,. \tag{B.21}$$

They operate in a similar fashion on $\Psi_{\{0,1\}}$ and indeed on the whole superfield. We can extend these shadow supersymmetries to operations that step up N steps

$$\mathcal{Q}^{[i]}_{a_1 a_2 \ldots a_N b_1 b_2 \ldots b_N} \equiv \mathcal{A}^\dagger_{a_1} \mathcal{A}^\dagger_{a_2} \ldots \mathcal{A}^\dagger_{a_N} \theta_i \mathcal{A}_{b_1} \mathcal{A}_{b_2} \ldots \mathcal{A}_{b_N}\,, \tag{B.22}$$

where $a_i, b_j = 0, 1, 2$, and $i = 1, 2$, and their inverses

$$\overline{\mathcal{Q}}^{[i]}_{a_1 a_2 \ldots a_N b_1 b_2 \ldots b_N} \equiv \mathcal{A}^\dagger_{a_1} \mathcal{A}^\dagger_{a_2} \ldots \mathcal{A}^\dagger_{a_N} \frac{\partial}{\partial \theta_i} \mathcal{A}_{b_1} \mathcal{A}_{b_2} \ldots \mathcal{A}_{b_N} \tag{B.23}$$

and similarly for the ones with $\bar{\mathcal{A}}$. The anticommutators between these operators will close to terms of the form $z_i \frac{\partial}{\partial z_i} P_j$.

We can also check the anticommutators

$$\left\{ \mathcal{Q}^{[i]}_{a_1 a_2 \ldots a_N b_1 b_2 \ldots b_N}, \overline{\mathcal{Q}}^{[i]}_{a_1 a_2 \ldots a_M b_1 b_2 \ldots b_M} \right\}$$

to see that they close to step operators. Al these operators generate an infinite superalgebra for which the superfield is a representation, although we have not been able to to write it in a compact form. It would be very important to find a finite subalgebra. However, there are some arguments against such an algebra since it would amount to the solve the same problem as the old one of finding supersymmetries with higher (half-integer) helicity operators than 1/2.

C. F_4 oscillator representations

Since F_4 has rank four, one expects all its representations to be generated by four [15] sets of oscillators transforming as **26**, but we find that only three suffice. We give the construction for compact and one non-compact case.

C.1 Compact case

We label each copy of 26 oscillators as $A_0^{[\kappa]}$, $A_i^{[\kappa]}$, $i = 1, \ldots, 9$, $B_a^{[\kappa]}$, $a = 1, \ldots, 16$, and their hermitean conjugates, and where $\kappa = 1, 2, 3$. Under SO(9), the $A_i^{[\kappa]}$ transform as **9**, $B_a^{[\kappa]}$ transform as **16**, and $A_0^{[\kappa]}$ is a scalar. They satisfy the commutation relations of ordinary harmonic oscillators

$$\left[A_i^{[\kappa]}, A_j^{[\kappa']\dagger} \right] = \delta_{ij} \delta^{[\kappa][\kappa']}\,, \qquad \left[A_0^{[\kappa]}, A_0^{[\kappa']\dagger} \right] = \delta^{[\kappa\kappa']}\,.$$

Note that the SO(9) spinor operators satisfy Bose-like commutation relations

$$\left[B_a^{[\kappa]}, B_b^{[\kappa']\dagger} \right] = \delta_{ab} \delta^{[\kappa][\kappa']}\,.$$

The generators T_{ij} and T_a

$$T_{ij} = -i \sum_{\kappa=1}^{4} \left\{ \left(A_i^{[\kappa]\dagger} A_j^{[\kappa]} - A_j^{[\kappa]\dagger]} A_i^{[\kappa]} \right) + \frac{1}{2} B^{[\kappa]\dagger} \gamma_{ij} B^{[\kappa]} \right\},$$

$$T_a = -\frac{i}{2} \sum_{\kappa=1}^{4} \left\{ (\gamma_i)^{ab} \left(A_i^{[\kappa]\dagger} B_b^{[\kappa]} - B_b^{[\kappa]\dagger} A_i^{[\kappa]} \right) - \sqrt{3} \left(B_a^{[\kappa]\dagger} A_0^{[\kappa]} - A_0^{[\kappa]\dagger} B_a^{[\kappa]} \right) \right\},$$

satisfy the F_4 algebra,

$$[T_{ij}, T_{kl}] = -i(\delta_{jk} T_{il} + \delta_{il} T_{jk} - \delta_{ik} T_{jl} - \delta_{jl} T_{ik}),$$
$$[T_{ij}, T_a] = \frac{i}{2} (\gamma_{ij})_{ab} T_b,$$
$$[T_a, T_b] = \frac{i}{2} (\gamma_{ij})_{ab} T_{ij},$$

so that the structure constants are given by

$$f_{ijab} = f_{abij} = \frac{1}{2} (\gamma_{ij})_{ab}.$$

The last commutator requires the Fierz-derived identity

$$\frac{1}{4} \theta \gamma^{ij} \theta \chi \gamma^{ij} \chi = 3\theta\chi\chi\theta + \theta\gamma^i \chi\chi\gamma^i \theta,$$

from which we deduce

$$3\delta^{ac}\delta^{db} + (\gamma^i)^{ac} (\gamma^i)^{db} - (a \leftrightarrow b) = \frac{1}{4} (\gamma^{ij})^{ab} (\gamma^{ij})^{cd}.$$

To satisfy these commutation relations, we have required both A_0 and B_a to obey Bose commutation relations (Curiously, if both are anticommuting, the F_4 algebra is still satisfied). One can just as easily use a coordinate representation of the oscillators by introducing real coordinates u_i which transform as transverse space vectors, u_0 as scalars, and ζ_a as space spinors which satisfy Bose commutation rules

$$A_i = \frac{1}{\sqrt{2}} (u_i + \partial_{u_i}), \qquad A_i^\dagger = \frac{1}{\sqrt{2}} (u_i - \partial_{u_i}),$$
$$B_a = \frac{1}{\sqrt{2}} (\zeta_a + \partial_{\zeta_a}), \qquad B_a^\dagger = \frac{1}{\sqrt{2}} (\zeta_a - \partial_{\zeta_a}),$$
$$A_0 = \frac{1}{\sqrt{2}} (u_0 + \partial_{u_0}), \qquad A_0^\dagger = \frac{1}{\sqrt{2}} (u_0 - \partial_{u_0}).$$

Using square brackets $[\cdots]$ to represent the Dynkin label of F_4, and round brackets $(\cdots)$ to represent those of SO(9), we list some of the combinations which will be used for investigating the solutions of Kostant's equation

$$u_1 + iu_2 \sim [0\,0\,0\,1] \sim (1\,0\,0\,0), \tag{C.1}$$
$$u_3 + iu_4 \sim [1\,0\,0\ -1] \sim (-1\,1\,0\,0), \tag{C.2}$$
$$\zeta_1 + i\zeta_9 \sim [0\,0\,1\ -1] \sim (0\,0\,0\,1), \tag{C.3}$$
$$\zeta_8 + i\zeta_{16} \sim [0\,1 -\,1\,0] \sim (0\,0\,1 -\,1), \tag{C.4}$$
$$\zeta_3 - i\zeta_{11} \sim [1\ -1\,1\,0] \sim (0\,1 -\,1\,1), \tag{C.5}$$
$$\zeta_6 - i\zeta_{14} \sim [1\,0\ -1\,1] \sim (0\,1\,0\ -1). \tag{C.6}$$

Hence $u_1 + iu_2$ and $\zeta_1 + i\zeta_9$ are the highest weights of the SO(9) representations $bf9, and$bf 16, respectively.

C.2 Non-compact case

With a slight modification we can use this representation for the non-compact form of the embedding of F_4 in $\mathrm{SO}(25,1)$.

The non-compact algebra $\mathrm{SO}(25,1)$ can be generated by changing the sign of one of the oscillator commutation relations, requiring that

$$\left[\mathcal{A}_M, \mathcal{A}_N^\dagger\right] = \eta_{MN}\,, \tag{C.7}$$

where

$$\eta_{00} = -1\,, \qquad \eta_{ij} = \delta_{ij}\,, \qquad \eta_{ab} = \delta_{ab}\,.$$

The non-compact F_4 subalgebra still has $\mathrm{SO}(9)$ as a compact subalgebra realized by the same hermitean generators

$$\widehat{T}_{ij} = -i\left(A_i^\dagger A_j - A_j^{\dagger]} A_i\right) - \frac{i}{2} B^\dagger \gamma^{ij} B\,.$$

The spinor generators are given by

$$\widehat{T}_a = -\frac{1}{\sqrt{2}}\left\{(\gamma_i)^{ab}\left(A_i^\dagger B_b - B_b^\dagger A_i\right) - i\sqrt{3}\left(B_a^\dagger A_0 - A_0^\dagger B_a\right)\right\}\,.$$

Note that they are no longer hermitean, still transform as $\mathrm{SO}(9)$ spinors, but obey the "wrong sign" commutator

$$\left[\widehat{T}_a, \widehat{T}_b\right] = -i(\gamma_{ij})_{ab}\,\widehat{T}_{ij}\,.$$

This corresponds to the particular non-compact form of F_4 with maximal $\mathrm{SO}(9)$ compact subgroup.

Note that these are not simply the compact form multiplied by i. Because of the minus sign in the A_0 commutator, the $B_a A_0^\dagger$ part of T_a gives a term like $B^\dagger B$ with the opposite sign from the compact case. This is good because we need to get a $\gamma^{ij}\gamma^{ij}$ term of the opposite sign to get the non-compact commutator, as these two are linked by the Fierz identity. This means that the $\gamma^i\gamma^i$ term must be the negative of the compact case; this is done by taking away the i that multiplies the $A^i B_b^\dagger(\gamma)^i_{ab}$ terms in T_a, leading to the form above.

References

[1] T. Pengpan and P. Ramond, *M(ysterious) patterns in* SO(9), *Phys. Rept.* **315** (1999) 137 [hep-th/9808190].

[2] T.L. Curtright, *Indices, triality and ultraviolet divergences for supersymmetric theories*, *Phys. Rev. Lett.* **48** (1982) 1704.

[3] L. Brink, A.M. Khan, P. Ramond and X.z. Xiong, *Continuous spin representations of the Poincaré and super-Poincaré groups*, hep-th/0205145.

[4] B. Gross, B. Kostant, P. Ramond and S. Sternberg, *The weyl character formula, the half spin representations, and equal rank subgroups*, *Proc. Nat. Acad. Sci.* **95** (1998) 8441 [math.RT/9808133].

[5] L. Brink and P. Ramond, *Dirac equations, light-cone supersymmetry, and superconformal algebras*, in *The many faces of the superworld*, M.A. Shifman ed., World Scientific, Singapore 2000, pp. 398–416.

[6] Lars Brink, *Euler multiplets, light-cone supersymmetry and superconformal algebras*, in proceedings of the *International conference on quantization, gauge theory, and strings: conference dedicated to the memory of professor Efim Fradkin*, Moscow 2000.

[7] P. Ramond, *Boson fermion confusion: the string path to supersymmetry*, *Nucl. Phys.* **101** *(Proc. Suppl.)* (2001) 45 [hep-th/0102012].

[8] B. Kostant, *A cubic Dirac operator and the emergence of euler number multiplets of representations for equal rank subgroups*, *Duke Math. J.* **100** (1999) 447.

[9] L. Durand III, *Phys. Rev.* **128** (1962) 434;
K. Case and S. Gasiorowicz, *Phys. Rev.* **125** (1962) 1055;
M. Grisaru, R. Pendleton and P. Van Nieuwhenhuizen, *Supergravity and the s matrix*, *Phys. Rev.* **D 15** (1977) 996;
B. de Wit and D.Z. Freedman, *Systematics of higher spin gauge fields*, *Phys. Rev.* **D 21** (1980) 358;
C. Aragone and S. Deser, *Consistency problems of spin-2 gravity coupling*, *Nuovo Cim.* **B57** (1980) 33;
S. Weinberg and E. Witten, *Limits on massless particles*, *Phys. Lett.* **B 96** (1980) 59.

[10] M.A. Vasiliev, *Progress in higher spin gauge theories*, hep-th/0104246.

[11] L. Brink, J.H. Schwarz and J. Scherk, *Supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 121** (1977) 77;
F. Gliozzi, J. Scherk and D.I. Olive, *Supersymmetry, supergravity theories and the dual spinor model*, *Nucl. Phys.* **B 122** (1977) 253.

[12] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, *Phys. Lett.* **B 76** (1978) 409.

[13] M. Atiyah, talk at the Swedish Royal Academy, Stockholm, September 2001

[14] V. Bargmann, *Rev. Mod. Phys.* **34** (1962) 829

[15] T. Fulton, *Realization of lie algebras and of representations of lie groups in terms of harmonic oscillators*, *J. Phys.* **A 18** (1985) 2863.